Readers' Forum

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Comment on "New Approach to Solution of the Falkner–Skan Equation"

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THE boundary conditions associated with ordinary differential equations of the boundary-layer type (Falkner–Skan equation) are as follows:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{1}$$

$$f = f' = 0, \qquad \eta = 0 \tag{2}$$

$$f' \to 1, \qquad \eta \to \infty \tag{3}$$

These equations are of the two-point asymptotic class. Here β is the pressure gradient parameter. Primes denote differentiation with respect to η. Specification of asymptotic boundary condition (3) implies that all higher derivatives of the dependent variable approach zero as the specified value of the independent variable is approached. Although Eqs. (1–3) do not contain η explicitly, it is impossible to integrate Eq. (1) with the asymptotic boundary conditions from the specified $\eta = \eta_{\max}$ downward to $\eta = \eta_0$, where $f = f' = 0$. When $f' \to 1$, $f'' \to 0$ and $(\eta - f)$ tends to a β-dependent constant in the asymptotic region $(\eta \geq \eta_{\max})$. It is possible to obtain $f''(\eta_0)$ by integrating Eq. (1) with the known initial conditions close to the asymptotic boundary from a specified $\eta = \eta_{\max}$ downward to $\eta = \eta_0$ (where $f = f' = 0$). By transformation of the results from η to $\eta - \eta_0$, the solution of boundary-layer equations (1–3) can be obtained.

Direct numerical integration requires that the two-point boundary-value problem be converted to an initial-value problem by estimating one of the three required initial conditions that satisfies the other boundary conditions. This two-point boundary-value problem poses a mathematical difficulty when one attempts to apply a simple shooting technique.

Sher and Yakhot[1] have recently proposed a new approach for integrating Eq. (1) from a specified $\eta = \eta_{\max}$ downward to $\eta = \eta_0$ (where $f' = 0$) using a Runge–Kutta method. They used initial conditions at $\eta = \eta_{\max}$ by specifying a value for f' (≥ 0.9), guessing a value for f, and evaluating f'' from the following equation:

$$f'' = \tfrac{1}{2}(1 - f')\left[f + \sqrt{f^2 + 4\beta(1 + f')}\right] \tag{4}$$

The value of f obtained at this point, $\eta = \eta_0$, is the corrector to the guess value of f (at $\eta = \eta_{\max}$) to obey the actual boundary condition ($f = 0$ at $\eta = \eta_0$). They used a predictor–corrector procedure for finding the required value of f at $\eta = \eta_{\max}$. Among the specified three values for $f' = 0.9$, 0.95, and 0.99 at $\eta = \eta_{\max}$, their numerical results show that specification of $f' = 0.99$ at $\eta = \eta_{\max}$ yields accurate values of $f''(0)$. Because Eq. (4) is obtained by applying L'Hopital's rule to the limit of the expression $(1 - f')/f''$ as $\eta \to \infty$, utilizing Eq. (1) and solving the resulting quadratic equation for positive f'', which is valid close to the asymptotic boundary. From the numerical results, they presented an approximate expression near the asymptotic boundary ($\eta = \eta_{\max}$): $f = 3.4044(\beta + 2)^{-0.58917}$, which can be used as the initial guess.

Following the Nachtsheim–Swigert iterative scheme,[2] Nataraja et al.[3] have presented a numerical procedure for the solution of boundary-layer equations. As such there is no mathematical difficulty in the technique of initially using a small value of $\eta_{\max}$ (say, 2) to obtain an approximately correct value for $f''(0)$ and then increasing $\eta_{\max}$ to obtain the final value of $f''(0)$. For the separation solution, the value of β (say, β_S) can be found easily by specifying $f''(0) = 0$ and satisfying the asymptotic boundary conditions. Specifying the negative values of $f''(0)$ and considering the value of $\eta_{\max}$ used to obtain the separation solution plus 2 as an initial estimate for $\eta_{\max}$, the value of β can be found for the reverse flow solutions by satisfying the asymptotic boundary conditions.[4]

For $\beta = 0$, solution of the problem, following Ref. 3, is obtained at $\eta = 3.5$: $f = 2.286409$, $f' = 0.990709$, and $f'' = 0.024415$, whereas at $\eta = 0$, $f'' = 0.469601$. Using these values at $\eta = 3.5$ as initial conditions and integrating Eq. (1) from $\eta = 3.5$ downward to $\eta = 0$, the following results are obtained at the end of integration: $f(0) = f'(0) = 0$ and $f''(0) = 0.469601$. Usage of the preceding initial conditions close to the asymptotic boundary resulted in the accurate solution through downward integration of the Falkner–Skan boundary-layer equation. With the preceding values of f and f' (at $\eta = 3.5$), Eq. (4) gives $f'' = 0.021243$, which is lower than the exact value of 0.024415. This indicates that Eq. (4) is valid only when f' is very close to unity. However, the guess value for f at the specified $\eta = \eta_{\max}$ can be improved by utilizing the following iterative process and satisfying the conditions $f = f' = 0$ at $\eta = \eta_0$.

For a specified value of the pressure gradient parameter β, Eq. (1) is integrated with the initial conditions at $\eta = \eta_{\max} = 4$:

$$f = f_g \tag{5}$$

(say, 5) and

$$f' = 0.999 \tag{6}$$

where f'' is obtained by substituting the preceding values for f and f' in Eq. (4). A fourth-order Runge–Kutta method is utilized for integrating differential equation (1) with a step size: $\Delta\eta = -0.1$. When f' becomes less than zero, the step size $\Delta\eta$ is reduced to half and differential equation (1) is solved with the obtained solution at

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Table 1 Comparative study on the solution of the Falkner–Skan equation

	Nachtsheim–Swigert iterative scheme					Modified new approach		
β	$f''(0)$	η	f	f'	f''	$(\eta_{\max} - \eta_0)$	$f(\eta_{\max})$	$f''(\eta_0)$
2	1.687211	2.7	2.202796	0.999143	0.003027	2.66508	2.16816	1.68733
1	1.232587	3.2	2.552326	0.999186	0.002756	3.15408	2.50684	1.23277
0	0.469601	4.3	3.083458	0.999152	0.002851	4.27649	3.06035	0.46978

the previous $\eta-$ station. In the present study, this process has been carried out until the step size $|\Delta\eta| < 0.00001$, at which $f' \to 0$. The values of η and f at this station are denoted by η_0 and f_0, respectively. If $f_0 = 0$, then the value of f'' at this station is the required solution. Otherwise, f_g has to be suitably modified and the process repeated until f_0 becomes zero. It is known that f is a monotonically increasing function from $\eta = 0$ to $\eta = \eta_{\max}$. When f_0 is greater than zero, f_g should be modified with a step size Δf_g (say, -0.1) and the process repeated until $f_0 \leq 0$. When f_0 becomes less than zero, the step size Δf_g should be reduced to half and the differential equation integrated with the previous f_g value plus Δf_g. This process is continued until $|\Delta f_g|$ reaches a value less than 0.0001. The value of f'' at $\eta = \eta_0$ is the required solution and $(\eta_{\max} - \eta_0)$ gives the value of η close to the asymptotic region in Eqs. (1–3) at which $f' = 0.999$.

Table 1 shows a good comparison of results using the modified new approach and the Nachtsheim–Swigert scheme. The discrepancy in the results of these approaches is mainly due to Eq. (4), which is valid very close to the asymptotic boundary. Both the approaches demand an iterative process and a guess value of the dependent variable to obtain the solution of the Falkner–Skan boundary-layer equation by direct integration. However, the numerical results of the present problem using the Nachtsheim–Swigert iterative scheme are found to be close to the exact literature values.[4,5]

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